Stochastic excitation and synchronization in coupled FitzHugh-Nagumo elements

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ABSTRACT

We investigate theoretically and numerically the activation process in a single-out and coupled FitzHugh-Nagumo elements. Two qualitatively different types of the dependence of the mean activation time and of the mean cycling time on the coupling strength monotonic and non-monotonic have been found for identical elements. The influence of coupling strength, noise intensity and firing threshold on the synchronization regimes and its characteristics is analyzed.

Keywords: FitzHugh-Nagumo system, stochastic activation, Fokker-Planck equation, optimal path

1. INTRODUCTION

Stochastic excitable dynamics of the type demonstrated by FitzHugh-Nagumo (FHN) elements underlies many systems in physiology and neuroscience.[1] In particular, a variety of nontrivial phenomena observed in coupled excitable systems perturbed by noise have attracted much attention recently. There are many resonance-like and synchronization-like effects in both small (two-element) and large (chains and lattices) ensembles of coupled bistable, excitable and oscillatory elements. Examples include e.g. stochastic[2–4] and coherence[5–10] resonance, stochastic synchronization,[11–14] and noise-induced and noise-enhanced propagation of fronts[15] of excitation.

However, the theory of noise-induced activation and synchronization in such far from equilibrium systems is still far from being complete.

In this work we investigate theoretically the activation rate of a single-out FHN element as a function of noise intensity and parameters of the driving force. Theoretical results are compared with the results of Monte Carlo simulations. This research allow us to gain further insight into the stochastic dynamics of synchronization of globally-coupled FNH elements. For the later case it has been found in numerical simulation that increasing the coupling strength can either increase or decrease mean activation times. Two qualitatively different types of the dependence of the mean activation time and of the mean cycling time on the coupling strength have been found for identical elements: (i) non-monotonic, when control parameters are close to the pint of Andronov-Hopf bifurcation and the noise intensity is relatively large, (ii) monotonic growth and saturation, when control parameters are away from the bifurcation point and the noise intensity is relatively small. For intermediate values of parameters the crossover between these two types of behavior takes place. For limiting cases of independent and ultimately strong coupled excitable elements dependence of mean activation time on noise intensity is accounted theoretically. By investigation of collective noise-induced dynamics in coupled non-identical FitzHugh-Nagumo elements synchronization-like phenomena of firing have been found. The influence of coupling strength, noise intensity and firing threshold on the synchronization regimes and its characteristics is analyzed.

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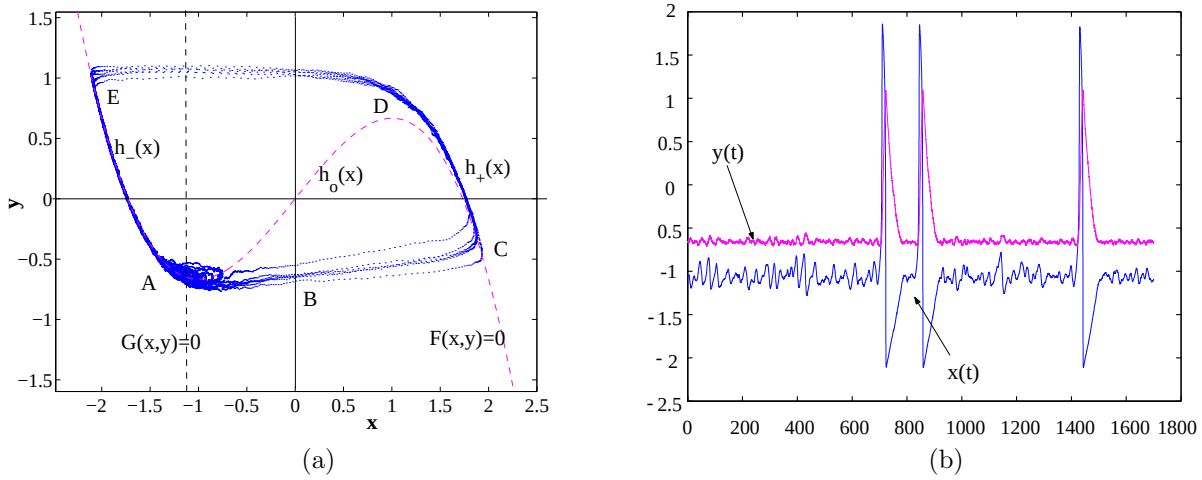
2. MODEL AND BASIC DEFINITIONS

Let us consider the noisy perturbed Fitz Hugh-Nagumo model described by the equations:

$$\begin{aligned} \dot{x} &= F(x,y) = & x - x^3/3 - y \\ \dot{y} &= \varepsilon G(x,y) + \sqrt{D}\zeta(t) = & \varepsilon(x+a) + \sqrt{D}\zeta(t), \end{aligned} \tag{1}$$

where $\zeta(t)$ is white gaussian noise with correlation function $< \zeta(t)\zeta(t+\tau) >= D\delta(\tau)$, ε is a small parameter, (see, for example[5, 16]). The curve $F(x,y) = 0$ has an N-like form, while $G(x,y) = 0$ is a strait line. There is only one steady sate of the system (1) corresponding to the point of intersection $A(x^0, y^0)$ of the curves $F(x,y) = 0$ and $G(x,y) = 0$ in zero-noise limit. This steady state has coordinates $x^0 = -a, y^0 = -a - a^3$. In the absence of noise all the trajectories of the system (1) will be attracted to $A(x^0, y^0)$.

In the presence of noise the state $A(x^0, y^0)$ becomes metastable, because relatively large fluctuations can kick the system away from the point A into the basin of attraction of the branch $h_+(x)$. The system then moves slowly along this branch, jumps back to the left branch and returns to the vicinity of steady state. The process of excitation repeats, and the resultant time series $x(t)$ has a sequence of noise-induced pulses of excitation.



(a) (b)

Figure 1. (a) Typical phase diagram for an excitable system, including the curves $F(x,y) = 0$ and $G(x,y) = 0$, and stochastic trajectories. Curves $h_-(x)$, $h_+(x)$, and $h_0(x)$ are left and right stable branches of slow motion, and unstable branch correspondingly. A, B, C, D, and E indicate characteristic points of the motion of the system in a course of excitation. (b) Random pulses of excitation x(t) and y(t)

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On the phase plane (x,y) we will distinguish two main types of motion: (i) activation, i.e. motion from the vicinity of the steady state across the line, separating basins of attraction of left $h_-(x)$ and right $h_+(x)$ stable branches of slow motion (what we shall name separatrix $y_s(x)$) towards the line $x = 0$ (approximately between points A and B); and (ii) excursion, i.e. motion after the first passage of the line $x = 0$ along the branches $h_+(x)$ and $h_-(x)$ back into the vicinity of the stable state (approximately from point B through points C,D,E, to point A). In our numerical experiments we will consider three characteristics timescales of this behaviour: the activation time t_A, the excursion time t_E, and the total time of the cycle ABCDE: $t_C = t_A + t_E$.

3. ACTIVATION TIME OF SINGLE FHN-ELEMENT

In the presence of noise the system dynamics is characterized* by the evolution of the probability density $\rho(x,t)$ described by the Fokker-Planck equation (FPE)[17]:

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x_i}\left(-\rho F_i + \frac{D}{2}\frac{\partial}{\partial x_j}Q_{ij}\rho\right), \tag{2}$$

where Q is the diffusion matrix. In the particular case of the 2-dimensional system (1) $F_1 = F(x,y)$, $F_2 = G(x,y)$, and $Q_{ij} = \delta_{i2}\delta_{j2}$. Since the FPE has the form of a continuity equation, with current density $J[\rho]$ equalling $\rho F_i - \frac{D}{2}\frac{\partial}{\partial x_j}Q_{ij}\rho$, the escape rate R may be computed as a flux of probability though the boundary $\partial\Omega$ of the basin of attraction Ω of a metastable state.

$$R \sim \int_{\partial\Omega} \mathbf{J}[\rho]\cdot \mathbf{n} dl \Big/ \int_{\Omega} \rho d\mathbf{x}, \tag{3}$$

where the of probability distribution density $\rho(x,y)$ in the case of small noise is often written as:

$$\rho(x,y) = z(x)\exp\{-S(x,y)/D\} \tag{4}$$

(see.[18,19] Here the action function $S(x)$ plays the role of a non-equilibrium potential[20] and $z(x)$ is a prefactor.

An asymptotic analysis shows[18–23] that to the leading order of approximation in D, function S satisfies a Hamilton-Jacobi equation for a classical action in the form $H(x,\nabla S) = 0$, where $H(x,p)$ is the Hamiltonian function equals $\frac{1}{2}Q_{ij}p_ip_j + K_i(x)p_i$.

The pattern of extreme trajectories and the action function along these trajectories are found by integrating Hamiltonian equations simultaneously with the equation for the action:

$$\dot{x}_i = K_i + Q_{ij}p_j \quad \dot{p}_i = -\frac{\partial K_j}{\partial x_i}p_j, \quad \dot{S} = \frac{1}{2}Q_{ij}p_ip_j. \tag{5}$$

with initial conditions (see e.g.[18]):

$$x_i(t_0) = x_i^{(0)}(t_0) + \delta x_i, \quad p_i(t_0) = S_{ij}^{(0)}\delta x_j, \quad S(t_0) = \frac{1}{2}S_{ij}^{(0)}\delta x_i\delta x_j. \tag{6}$$

For a particular case of the system (1) $\vec{x}^{(0)} = \{x^{(0)}, y^{(0)}\}$ are coordinates of the stable state A, where the function $S(\vec{x})$ reaches its minimum. $S_{ij}^{(0)} = S_{ij}^{(0)}\Big|_{\vec{x}=\vec{x}^{(0)}}$ is the value of the Hessian matrix $S_{ij} = (\partial_i\partial_j S)$ in the point $\vec{x}^{(0)}$, and δx_i is a small deviation in x_i-direction from the stable state.

The set of trajectories solutions of (5) emanating from the stationary state of the system (1) covers its phase space. To obtain $S_{ij}^{(0)}$ we have taken into account, that near the stable state the stationary probability distribution (4) takes the form ([17,21,24]):

$$\rho(x,t) \cong \sqrt{\det(S^{(0)})/(2\pi D)^n}\exp\left(-\frac{1}{2D}S_{ij}^{(0)}\delta x_i\delta x_j\right). \tag{7}$$

On linearization F_i near the stable state the corresponding term in the Fokker-Planck equation (2) takes the form

$$\frac{\partial}{\partial x_i}(\rho F_i) = B_{ij}\frac{\partial \rho}{\partial x_i}x_j, \quad \text{where} \quad B_{ij} = \frac{\partial F_i}{\partial x_j}\Big|_{\vec{x}=\vec{x}^{(0)}} \tag{8}$$

*Here we neglect the time of motion from the separatrix to the line $x = 0$, which is much smaller compered to activation time exponentially growing with noise intensity decreasing.

On substituting (7) into the linearized Fokker-Planck equation (see (2),(8) we have (see e.g.[25])

$$\dot{W} = -BW - WB^T - WQW, \tag{9}$$

where $W = (S^{(0)}), \quad B = (B_{ij})$.

Equation (7) can also be written in the form[17, 21, 24]:

$$\rho(\vec{q}) = const \exp\left\{-\frac{D_y\,\delta x^2 - 2\,B_{xy}\,\delta x\,\delta y + D_x\,\delta y^2}{2\,\left(D_x\,D_y - B_{xy}^2\right)}\right\}. \tag{10}$$

Therefore by solving equation (9) we obtain values of the components of covariation matrix of Gaussian distribution near the stable state (neglecting the possibility of escape from the neighborhood of these stable state):

$$D_x = \frac{1}{2}\,\frac{D}{\varepsilon\,(a^2-1)}, \quad B_{xy} = -D/(2\,\varepsilon)\,, \qquad D_y = \frac{1}{2}\,\frac{D\,((a^2-1)^2+\varepsilon)}{\varepsilon\,(a^2-1)} \tag{11}$$

Let us introduce parameter

$$\mu = \sqrt{2}\sigma_y/r, \tag{12}$$

where $\sigma_y = \sqrt{D_y}$ and $r \cong \max\{\varepsilon, (a-1)^2\}$ is the distance from the steady state to the separatrix $y_s(x)$. Following[18, 24, 26] in the next-to leading order of approximation in D of the solution of Fokker-Planck equation we can obtain equations for the prefactor z and Hessian matrix $S_{ij} = (\partial_i\partial_j S)$:

$$\begin{aligned} \frac{dz}{dt} &= z\left(\partial_i K_i + \frac{1}{2}Q_{ij}S_{ij}\right) \\ \dot{S}_{ij} &= -p_m\partial_i\partial_j K_m - S_{im}\partial_j K_m - S_{jm}\partial_i K_m - S_{jm}S_{ik}Q_{km} \end{aligned} \tag{13}$$

We mention finally that the mean activation time can now be represented in the form:

$$T_{act} \;=\; \langle t_A\rangle \;=\; T_0\,\exp\{S_m(D)/D\}\,, \tag{14}$$

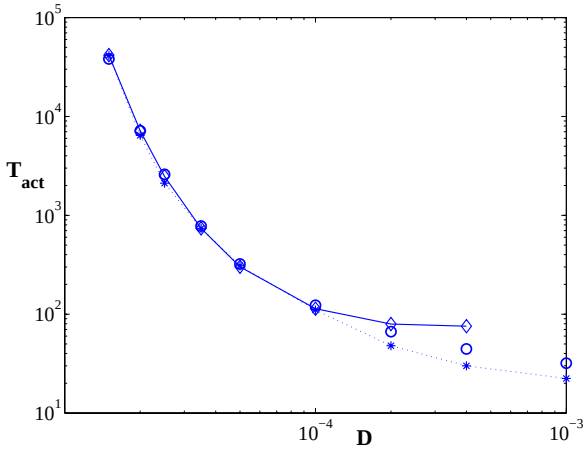

Figure 2. The dependence of activation times on the noise intensity in the small- and intermediate-μ (small- and intermediate- noise) cases: o - results of numerical simulation and averaging over 5000 activation escapes; — and diamonds - theoretical prediction using (3) and probability current integrating over the boundary; ...- theoretical prediction using (14).

where $S_m(D)$ is a minimum value of the quasipotential on the separatrix in the presence of finite noise and T_0 is a prefactor having dimension of time. It should be mentioned that due to finite noise corrections (in contrast to the case of gradient field) both prefactor and the value $S_m(D)$ depend on the noise intensity D.

The theoretical prediction (3), (14) for the value of $T_{act}(D)$ are compared with the results of numerical simulation for singled-out FHN element in Fig. 2. For this comparsion we have chosen the following values of parameters: $a = 1.05, \varepsilon = 0.05$, $D = 3.5 \cdot 10^{-5}$. It can be see that both methods give good qualitative estimation for $\mu \lesssim 1$ and are in good quantitative agreement for $\mu \lesssim 0.5$

4. INFLUENCE OF COUPLING ON STATISTICAL CHARACTERISTICS OF ACTIVATION AND CYCLE TIMES

In this section we analyze cooperative dynamics, namely the synchronization effects, in the system of two coupled FHN elements:

$$\begin{aligned} \dot{x}_1 &= x_1 - x_1^3/3 - y_1 + C(x_2 - x_1), \\ \dot{y}_1 &= \varepsilon\ (y_1 + a_1) + \zeta_1(t), \\ \dot{x}_2 &= x_2 - x_2^3/3 - y_2 + C(x_1 - x_2), \\ \dot{y}_2 &= \varepsilon\ (y_2 + a_2) + \zeta_2(t), \end{aligned} \tag{15}$$

where a_i are parameters of the partial element; C is the coupling coefficient; $\zeta_{1,2}(t)$ are the independent components of a two-dimensional Gaussian white noise with correlation functions $\langle \zeta_i(0)\zeta_j(t)\rangle = D\delta_{ij}\delta(t)$. In our numerical experiments we take both a_i slightly larger than one, $\varepsilon = 0.05$. In this section we will use simulation to obtain as a function of the coupling strength the mean activation time T_{act}, and the mean cycle time T_{cycl}. By varying the value of μ, three different types of behavior can be distinguished. Therefore, we will consider three intervals of μ: small μ, large μ and intermediate μ.

4.1. Small activation energy ($\mu > 1$)

In this case we take the parameter a near to the bifurcation point (i.e. close to one) and relatively large noise intensity D such that μ is bigger then unity. The results of the simulation of activation time T_{act} and time of the full cycle T_{cycl} are shown in Figure3(a). Both T_{act} and T_{cycl} of subsystems initially decrease and subsequently increase to reach constant values in the absence of coupling.

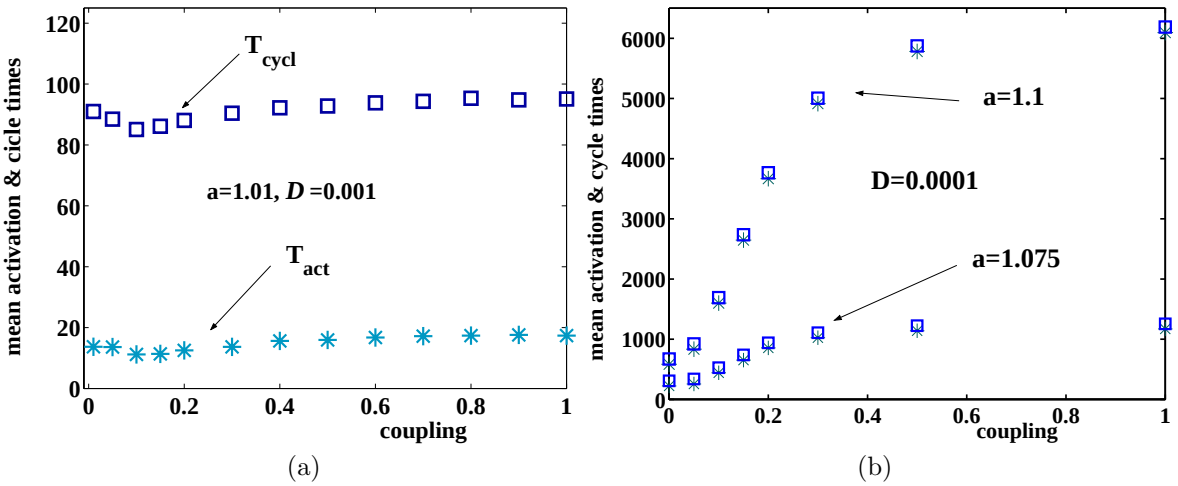

Figure 3. (a) The T_{cycl} (squares) and T_{act} (stars) for two coupled identical FHN-elements with parameters $a = 1.01$, $D = 0.001$. (b) The same quantities with $D = 0.0001$, $a = 1.075$, and $a = 1.1$.

To explain this result we note that the activation process is analogous to the escape over a barrier from a potential well. For the chosen parameter values there is almost no barrier, and we observe not activation but diffusion through the boundary. Therefore the activation time is small in comparison to the excursion time. The small decreasing of T_{act} and T_{cycl} occur because one starting the activation process FHN-element raise the probability of activation for an other element.

4.2. Large activation energy ($\mu \ll 1$)

In this case the parameter a is not very close to one, and the noise intensity is very small. Simulation results are shown in Fig. 3(b). The behavior of the system is qualitatively different from the previous case: both T_{act} and T_{cycl} increase monotonically with increasing coupling strength of elements, and $T_{cycl} \approx T_{act}$.

It can be seen from Fig. 3(b) that T_{act} initially increases quasiexponentially and then come to saturation. The level of saturation for $a = 1.075$ is six times larger then the initial value (in the absence the coupling); for $a = 1.1$ this level is twelve times larger then the initial value.

Because in this case the activation time is much larger than the excursion time, the system spends most of its time near the stable state. We therefore assume that both elements are initially situated in the neighbourhood of the stable state. If the coupling is extremely small the activation processes from the neighbourhood of the stable states of each FHN element can be considered as independent. But already for an intermediate strength of coupling the existence of coupled element hampers the escape and leads to a significant increase of T_{act}. We can say that in this case there are two counteracting tendencies. (i) If both elements are deactivated the T_{act} of each element is increased due to the coupling. (ii) After one of the elements has been activated the T_{act} of the second element is decreased due to the coupling (as compared to the T_{act} of uncoupled element). For the values of coupling considered in this subsection the first tendency is the stronger. An increase of coupling strength will eventually lead to the synchronized activation of both elements.

4.3. Intermediate activation energy

For intermediate values of the control parameter, the T_{act} initially decreases with increasing coupling strength, and then increases considerably and becomes $\sim 1.2 - 3.0$ times larger than the T_{act} value for uncoupled element.

4.4. Theoretical explanation

To estimate the maximal growth the T_{act} due to the coupling, we should bear in mind that, in the case of small noise the activation time being the special case of the mean escape time T_{esc} from the basin of attraction of stable state can often be presented in the form (see section (3), equation (14)):

$$T_{act}^{single} = T_{esc} = T_0 \exp\left\{\frac{S(D)}{D}\right\}, \tag{16}$$

where T_0 is a co-factor independent of (or only weakly dependent on) the noise intensity, $S(D)$ is the quasipotential or action for the optimal escape trajectory going in the point corresponding the maximum of transverse component of probability density current $\mathbf{J}[\rho]$ across the separatrix. To take account of corrections induced by the non-zero noise intensity, we should calculate probability density current $\mathbf{J}[\rho]$ using the approach suggested in[24] and articulated shortly in section (3).

We can also obtain the analogous result for a single element with variables $x_+ = (x_1+x_2)/2$ and $y_+ = (y_1+y_2)/2$ defined by equations

$$\begin{aligned} \dot{x}_+ &= x_+ - \frac{1}{3}\,{x_+}^3 - y_+ \\ \dot{y}_+ &= \varepsilon\,(x_+ + a_+ - g\,y_+) \quad + \quad \zeta_+(t)\ , \end{aligned} \tag{17}$$

with noise intensity $D_+ = D/2$. Then

$$T_{act}^{coupl} = T_0 \exp\left\{\frac{2S(D/2)}{D}\right\}. \tag{18}$$

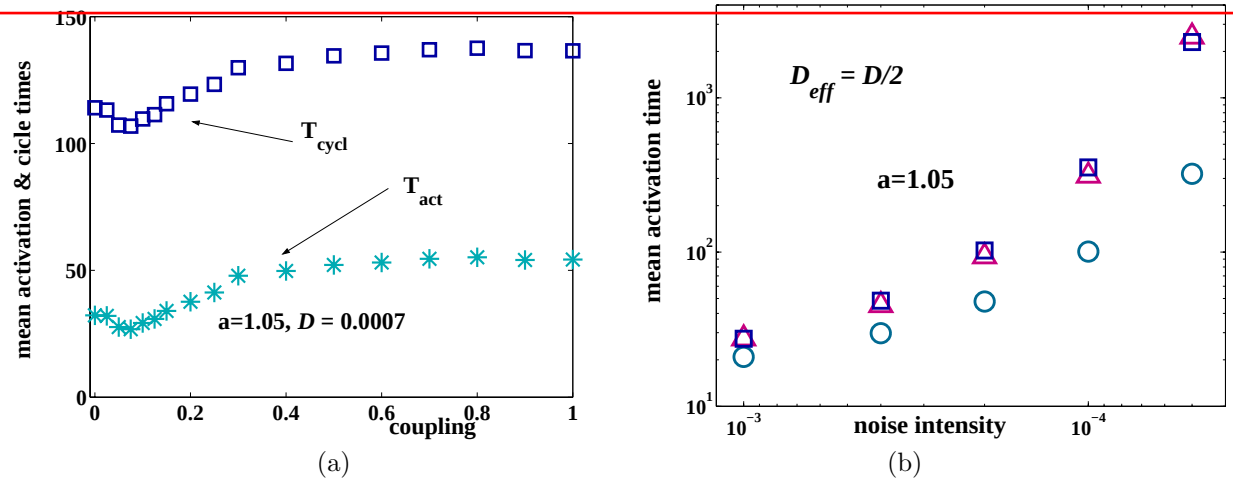

Figure 4. (a) The left part shows the T_{cycl} (squares) and T_{act} (stars) for two coupled identical FHN-elements with parameters $a = 1.05$, $D = 0.0007$. (b) The right part shows the activation time versus noise intensity: numerical simulation data for T_{act}^{single} (circles) and T_{act}^{coupl} (squares) are compared with theoretical estimates of $T_{act.estim}^{coupl}$ (triangles).

Using (16) and (18), our estimation for the mean activation time of the strongly coupled elements can finally be written as:

$$T_{act.estim}^{coupl} = T_{act}^{single} \exp\left\{\frac{2S(D/2) - S(D)}{D}\right\}. \tag{19}$$

In Fig. 4(b) we compare the dependencies of T_{act}^{single}, $T_{act.estim}^{coupl}$ with experimentally obtained values T_{act}^{coupl} in strong coupling case $C = 10$ for $a = 1.05$. Given that $2\,S(D/2) - S(D) \simeq 2\,S(D) - S(D) = S(D)$, one can see that the relation (19) can be used to describe the influence on the T_{act} and T_{cycl} of all the values and types of coupling discussed: for $S(D)/D \ll 1$ we obtain the case of large μ; for $S(D)/D > 1$ - the case of small μ; and the case of intermediate μ for $S(D)/D \simeq 1$.

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