Oscillatory and rotatory synchronization of chaotic autonomous phase systems

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The existence of rotatory, oscillatory, and oscillatory-rotatory synchronization of two coupled chaotic phase systems is demonstrated in the paper. We find four types of transition to phase synchronization depending on coherence properties of motions, characterized by phase variable diffusion. When diffusion is small the onset of phase synchronization is accompanied by a change in the Lyapunov spectrum; one of the zero Lyapunov exponents becomes negative shortly before this onset. If the diffusion of the phase variable is strong then phase synchronization and generalized synchronization, occur simultaneously, i.e., one of the positive Lyapunov exponents becomes negative, or generalized synchronization even sets in before phase synchronization. For intermediate diffusion the phase synchronization appears via interior crisis of the hyperchaotic set. Soft and hard transitions to phase synchronization are discussed.

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I. INTRODUCTION

Synchronization of chaotic oscillations is a fundamental phenomenon observed in nature and science. Three main types of synchronization have been studied, namely, complete (or full) synchronization [1], generalized synchronization [2], and phase synchronization [3] (for a review about chaotic synchronization, see Refs. [3–6]). Complete synchronization of identical systems occurs when the states of coupled systems coincide; the coupling should be strong enough to suppress the chaotic instability and to make one of the positive Lyapunov exponents negative. A similar situation, in the sense of a change of the Lyapunov exponents spectrum, usually take place by generalized synchronization of coupled nonidentical oscillators. Contrary to complete and generalized synchronization, the phase locking can appear for relatively small coupling when all positive Lyapunov exponents remain positive.

Chaotic phase synchronization of coupled oscillators, first demonstrated for paradigmatic dynamical models, the Rössler and Lorenz systems [7–12], has been observed in many laboratory and natural systems [13]. This type of chaotic synchronization is very similar to the synchronization of periodic oscillators and is manifested in the occurrence of locking between suitably defined phases, while the amplitudes remain nearly uncorrelated. Recently, phase synchronization of chaotic rotators has been studied for coupled *non-autonomous continuous-time* rotators and for *discrete-time* rotators, i.e., the circle maps [14–16]. It has been found that phase synchronization occurs via a crisis transition [17] to a band-structured chaotic attractor. At that the Lyapunov exponents corresponding to both phase variables remain positive. It is important to note that, in general, there is no zero Lyapunov exponent in these systems in the chaotic regime. In this paper we study synchronization phenomena in coupled *autonomous continuous-time phase systems* [18].

The paper is organized as follows. In Sec. II we describe the model under study, present two of its main properties, and introduce two types of chaotic phase synchronization. In Secs. III–V we present our numerical results of synchronization of rotatory, oscillatory, and oscillatory-rotatory phase variables, respectively. Section VI is devoted to describe hard and soft transitions to phase synchronization. The results are summarized in Sec. VII.

II. MODEL

The uncoupled model system is described by the following equation:

$$\mu\frac{d^3\phi}{dt^3}+\frac{d^2\phi}{dt^2}+\frac{d\phi}{dt}+\sin\phi=\gamma, \tag{1}$$

where ϕ is the phase variable defined in the interval $[-\pi,\pi]$, and μ and γ are non-negative parameters. Model (1) is not only a paradigmatic model that we use to show some nontrivial synchronization effects, but it is also a model of a Josephson junction with constant biased current and subject to a load with inductance, resistance, and capacitance [19]. Model (1) is also a model of a phase-locked loop (PLL) system with the simplest second-order filter [21]. These standard PLL circuits, well known in radio engineering, can operate in the regime of generation of chaotically modulated signals with the carrier stabilized at a reference frequency.

The following two properties of Eq. (1) are important to study peculiar synchronization processes in coupled systems: (i) one of the variables is the phase variable and (ii) chaos possesses zero Lyapunov exponent, i.e. in the chaotic parameter regime, the dynamics has a zero Lyapunov exponent. Due to the first property we will distinguish two types of chaotic phase synchronization: (i) "real" chaotic phase synchronization (RCPS) and (ii) generalized chaotic phase synchronization (GCPS). In the case of RCPS the well-known conditions of phase and frequency locking of two coupled systems should be fulfilled [22] and hyperchaos, i.e., the existence of two positive Lyapunov exponents, should take place. In the case of GCPS only one Lyapunov exponent remains positive, although the phase and frequency locking

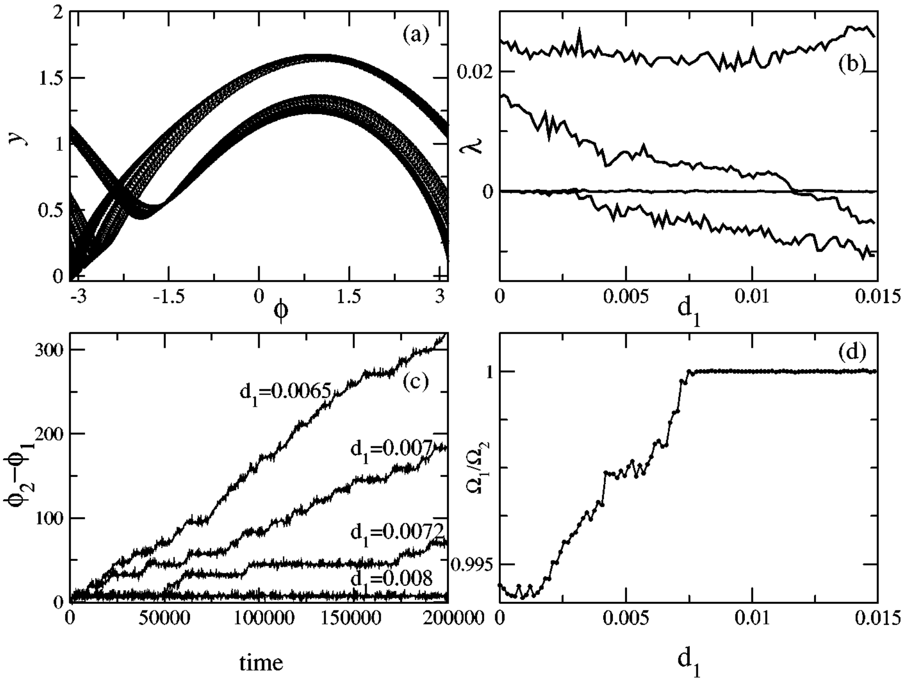

FIG. 1. Synchronization of rotatory phase variables. (a) Projections of the typical rotatory trajectory of the system (1) on the (ϕ,y) plane. Parameters are $\gamma=0.645$, $\mu=3.0$. In (b)–(d) parameters are $\gamma_1=0.645$, $\gamma_2=0.667$, $\mu=3.0$, and $d_2=0$. (b) The four largest Lyapunov exponents, one of which is always zero. (c) Difference of phase variables $\phi_2-\phi_1$ for nonsynchronous $(d_1=0.0065;0.007;0.0072)$ and synchronous $(d_1=0.008)$ regimes. (d) The mean frequency ratio Ω_1/Ω_2 vs coupling.

conditions are fulfilled. There is another type of synchronization, generalized chaotic synchronization (GCS), at which only one Lyapunov exponent is positive but phase and/or frequency locking does not take place. It is very important to note that the negativity of the Lyapunov exponents is only a necessary condition for the stability of the synchronous state. But very often [23] the transition to GCS is rather close to the transition of one of the Lyapunov exponents from positive to negative values. Therefore, we will conclude the onset of GCS when one of the positive Lyapunov exponents becomes negative.

Due to the second property, that chaos possesses a zero Lyapunov exponent, there are many properties in common between phase synchronization of autonomous chaotic oscillators and phase synchronization of autonomous chaotic phase systems.

In order to study synchronization phenomena in coupled nonidentical chaotic phase systems (1), we consider the following model equations:

$$\dot{\phi}_{1,2}=y_{1,2},$$

$$\dot{y}_{1,2}=z_{1,2},$$

$$\mu_{1,2}\dot{z}_{1,2}=\gamma_{1,2}-\sin\phi_{1,2}-y_{1,2}-z_{1,2}+d_1(y_{2,1}-y_{1,2})+d_2(z_{2,1}-z_{1,2}), \tag{2}$$

where $d_{1,2}$ are the coupling coefficients.

Depending on the parameter values the uncoupled system can demonstrate three types of chaotic behavior [24]: (i) rotations, (ii) oscillations, and (iii) oscillations-rotations. We will investigate synchronization phenomena for all those types of chaotic dynamics. The effect of synchronization of chaos realized in a system of two coupled PLL generating chaotic signals can be used in secure communication applications. Such important properties of PLL as high accuracy of synchronization and the possibility of very simple control make the PLL very promising for data communication using not only regular but chaotic signals as well [25]. Unidirectionally coupled chaotic PLLs analogous to Eq. (1) have been considered in Ref. [26,27]. In [26] chaotic phase synchronization and in Ref. [27] almost complete chaotic synchronization are presented.

As well as for periodic synchronization, the appearance of chaotic phase synchronization is affected by the frequency mismatch of the coupled subsystems and by the coherence property of the motions. We will characterize this property, i.e., the diffusion of the phase variables, by their variances $D_{\phi_{1,2}}$ that are defined for large times as

$$\langle(\dot{\phi}_{1,2}-\langle\dot{\phi}_{1,2}\rangle)^2\rangle=D_{\phi_{1,2}}, \tag{3}$$

where $\langle\cdot\rangle$ is time averaging. We will show below that these variances $D_{\phi_{1,2}}$ of both coupled subsystems (as well as their frequency mismatch) play a crucial role in the transitions to phase synchronization.

III. PHASE SYNCHRONIZATION OF ROTATORY PHASE VARIABLES

In this case, phase variables $\phi_{1,2}$ unboundedly increase and $\dot{\phi}_{1,2}$ are always (or almost always) positive. A projection of the chaotic phase rotating trajectory on the (ϕ,y) plane [Fig. 1(a)] looks like a "smeared" periodic trajectory with monotonically (or almost always) increasing phase. Therefore, the phase synchronization of chaotic rotations is quite similar to the periodic synchronization, i.e. in both cases only the phase growth rate is important. The averaged growth rate of phases or the mean frequency of rotations can be defined as

$$\Omega = \langle \dot{\phi} \rangle = \langle y \rangle. \tag{4}$$

In order to test for the existence of phase synchronization [31], we use two criteria. A chaotic synchronization of the rotations occurs if the mean frequencies characterizing the long time scale behavior of the coupled systems become equal:

$$\Omega_1 = \Omega_2. \tag{5}$$

On the short time scale, i.e., inside the $[-\pi;\pi]$ interval, due to the high diffusion of the phases, the transient phase differences can be rather large. The second criterion we use is the phase locking criterion

$$|\phi_2(t) - \phi_1(t)| \leqslant \text{const} \tag{6}$$

that ignores the short time scale behavior as well. Phase synchronization according to criteria (5) and (6) can be observed for systems, where the evolution of the phase variables behaves as an alternation of large intervals (where the phase variable increases) with relatively small intervals (where the phase variable decreases). We will demonstrate the existence of both the types of phase synchronization: RCPS and GCPS for such a type of behavior.

In our calculations, we set $\gamma_1 = 0.645$, $\gamma_2 = 0.667$, $\mu_{1,2} = 3.0$, and $d_2 = 0$. For these parameters the diffusion of phases is relatively large in both systems $(D_{\phi_1} \approx 0.219, D_{\phi_2} \approx 0.216)$, which affects the occurrence of phase synchronization. To illustrate the corresponding transition to phase synchronization, we plot the four largest Lyapunov exponents [Fig. 1(b)] and the mean frequency ratio [Fig. 1(d)] versus coupling, as well as the difference between the phase variables $\phi_1 - \phi_2$ for different couplings strength [Fig. 1(c)]. One can see that the real phase synchronization occurs at $d_1^1 \approx 0.0076$ [Fig. 1(d)]. For $d_1 > d_1^1$, the frequency and phase locking conditions (5) and (6) are satisfied, but hyperchaotic attractor still exists.

It is known [9] that for phase-coherent attractors phase synchronization sets in approximately at that value of coupling when one of the zero Lyapunov exponents becomes negative. In our simulations we find [Fig. 1(b)] that one of the zero Lyapunov exponents becomes negative already at $d_1 \approx 0.003$. But the transition to RCPS in system (2) occurs for essentially larger coupling. The occurrence of phase synchronization takes place via a crisis transition of the structure of the hyperchaotic attractor, i.e. via an *interior crisis* of the chaotic set.

At larger coupling $(d_1^2 \approx 0.0118)$, where one of the positive Lyapunov exponents becomes negative, GCPS occurs. Due to the relatively high noncoherence properties, the interval of the values of coupling between the transitions to RCPS and to GCPS $L = [d_1^1; d_1^2]$ is small. As our numerical simulations show, the increase in the parameters $\gamma_{1,2}$ leads to a complication of the topological structure of the chaotic attractors. The intervals, where the phase variables decrease, become larger and the behavior transfers from a rotational type to an oscillation-rotational one. This leads to an increase in the noncoherence properties of motion (diffusion of the phase variable increases) and as a result of that the width of the L interval between RCPS and GCPS tends to zero. The reason for that is the following. The chaotic phase synchronization is similar to the synchronization of periodic oscillations in the presence of noise [9]. When noise increases, a larger coupling is needed to achieve phase locking. By analogy, in order to suppress large phase fluctuations by chaotic phase synchronization, a stronger coupling has to be applied.

IV. PHASE SYNCHRONIZATION OF OSCILLATORY PHASE VARIABLES

In this case in both subsystems in Eq. (2) the phase variable oscillates around some constant value (i.e., $\phi_{1,2}$ are bounded) [Fig. 2(a)]. Synchronization of such oscillatory phase variables is quite similar to the case of usual phase synchronization of chaotic oscillators [9]. Because of the simple topology of the chaotic attractor one can introduce a new "artificial" phase variable

$$\psi = \arctan \frac{y}{\phi - \arcsin \gamma}, \tag{7}$$

a new amplitude

$$A = [(\phi - \arcsin \gamma)^2 + y^2]^{1/2}, \tag{8}$$

and the mean frequency:

$$\omega = \langle \dot{\psi} \rangle = \lim_{T \to \infty} \frac{\psi(T) - \psi(0)}{T}. \tag{9}$$

Here conditions (5) and (6) have been applied to the new phase variables $\psi_{1,2}$ and the mean frequencies $\omega_{1,2}$ can be used as criteria of synchronization. Therefore, although the oscillatory and rotatory cases cannot be generally reduced one to another, two similar criteria of the existence of phase synchronization can be used and as we will show, many similar effects take place. For the chosen parameters $\gamma_1 = 0.815$, $\gamma_2 = 0.83$, and $\mu_{1,2} = 3.3$, the coherence of motions is rather high $(D_{\psi_1} \approx 0.075, D_{\psi_2} \approx 0.079)$. We consider y and z couplings [in Eq. (2) $d_1 = d_2 = d$]. As in the case of phase synchronization of rotatory phase variables, we compute the Lyapunov spectrum [Fig. 2(b)], the frequency ratio [Fig. 2(d)], and the evolution of the phase variable difference [Fig. 2(c)]. For oscillatory phase variables both phase synchronizations, RCPS and GCPS, are found. With an increase in the coupling, the frequency ratio $\rho = \omega_2/\omega_1$ decreases to 1 smoothly (without any jump), i.e., a soft transition to RCPS takes place. This is manifested in the evolution of the phase variable difference, namely for a coupling close to the critical value $d^1 = 0.0082$, phase locking at large time intervals is observed [Fig. 2(c)]. Due to the high coherence of motions, i.e., small phase diffusion, phase locking and frequency entrainment occur approximately (shortly after) at the same value of coupling for which one of the zero Lyapunov exponents becomes negative. It should be mentioned that by the transition to synchronization of the "artificial" phases $\psi_{1,2}$ the new amplitudes $A_{1,2}$ as well as the real phases $\phi_{1,2}$ re-

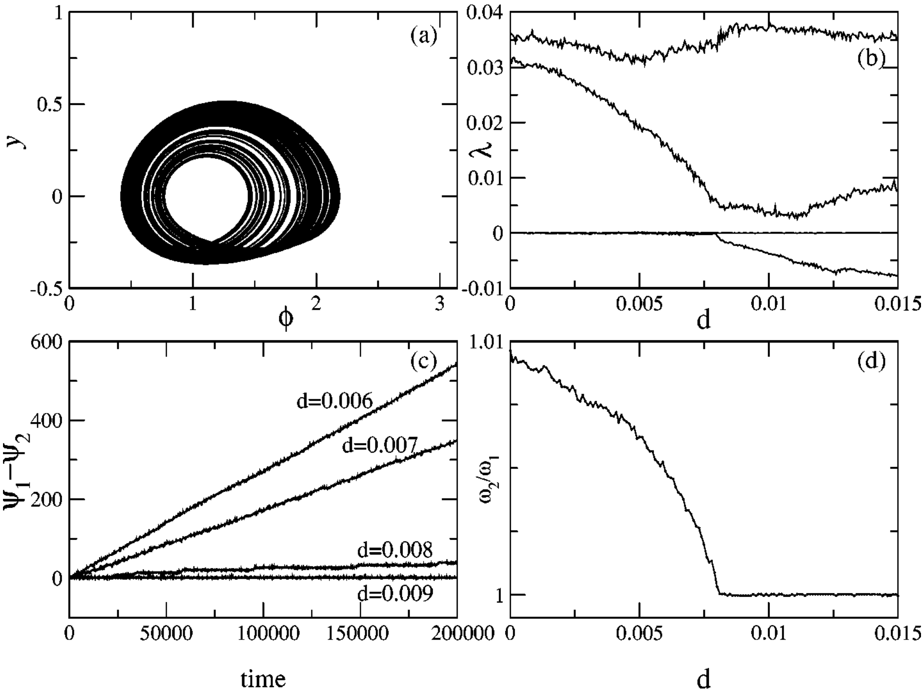

FIG. 2. Synchronization of oscillatory phase variables. (a) Projections of a typical oscillatory trajectory of system (1) on the (ϕ,y) plane for the parameters $\gamma=0.83$, $\mu=3.3$. In (b)–(d) the parameters are $\gamma_1=0.815$, $\gamma_2=0.83$, and $\mu_{1,2}=3.3$. (b) The four largest Lyapunov exponents. (c) Difference of phase variables $\psi_1-\psi_2$ of the y- and z-coupled subsystems in Eq. (2) $(d_1=d_2=d)$ for nonsynchronous $(d=0.006;0.007)$, nearly synchronous $(d=0.008)$, and synchronous $(d=0.009)$ regimes. (d) The mean frequency ratio ω_1/ω_2 vs coupling.

main highly uncorrelated (Fig. 3). But some frequency entrainment sets in. The averaged number of oscillations per unit time, computed easily as the number of maxima, coincide for both phases for $d>d^1$.

At essentially larger coupling $(d^2=0.043)$ generalized phase synchronization and as a result a strong correlation of all variables arise. Interval $L=[d_1^1;d_1^2]$ between the transitions to RCPS and to GCPS is relatively large. As in the case of rotatory synchronization, we observe that when the noncoherence properties increase with an increase in the $\gamma_{1,2}$ parameters, the L interval becomes smaller and tends to zero.

V. PHASE SYNCHRONIZATION OF OSCILLATORY-ROTATORY PHASE VARIABLES

A quite different situation occurs in the case of oscillatory-rotatory behavior of phase variables [Fig. 4(a)]. The existence of phase synchronization, is in general, a nontrivial effect because the phase variables $\phi_{1,2}$ increase nonmonotonically. Their evolution is an alternation between time intervals (where the phase variable increases) and time intervals (where the phase variable decreases). Due to the similar lengths of both intervals, it is impossible to separate the evolution of the phase variables into two different time

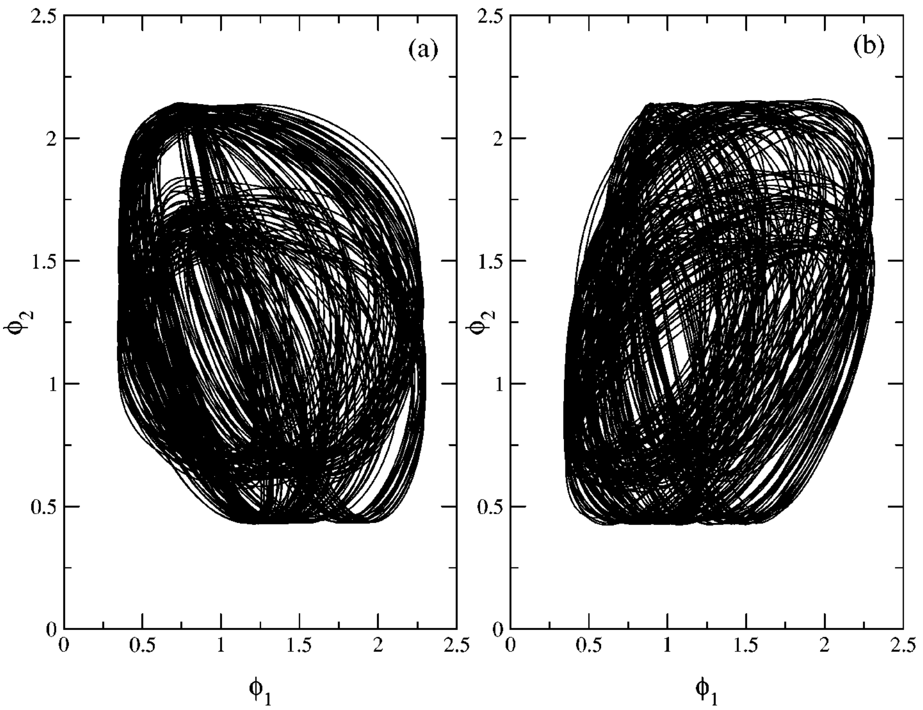

FIG. 3. Projections of the trajectories of system (2) on the (ϕ_1,ϕ_2) plane outside the synchronization region (a) $(d=0.008)$, and within the synchronization region (b) $(d=0.009)$. Parameters are $\gamma_1=0.815$, $\gamma_2=0.83$, and $\mu_{1,2}=3.3$.

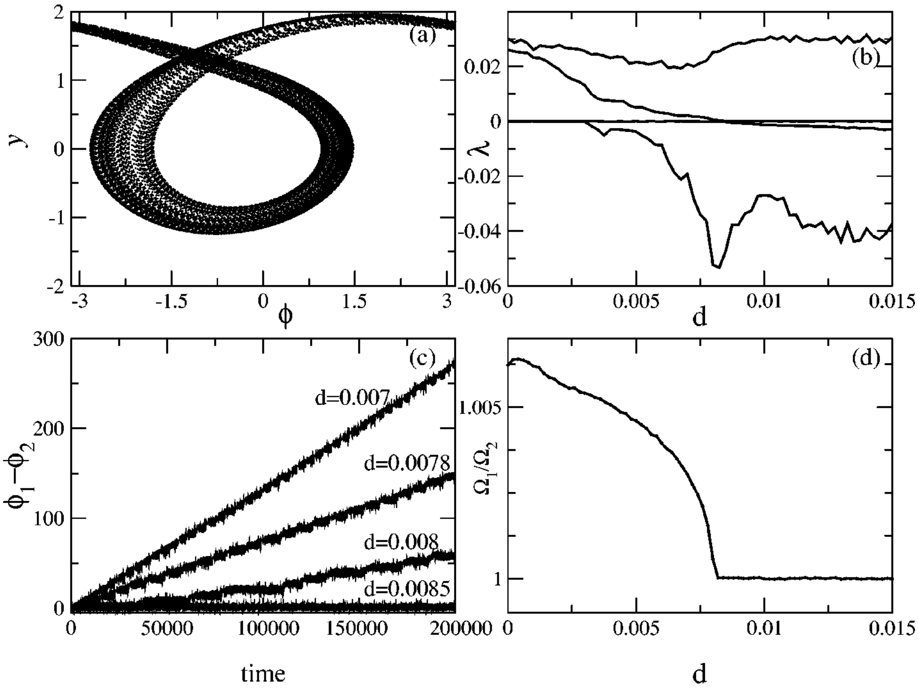

FIG. 4. Synchronization of oscillatory-rotatory phase variables. GCPS and GCS occur practically simultaneously at $d \approx 0.0082$. (a) Projections of the trajectory of system (1) on the (ϕ, y) plane. Parameters are $\gamma = 0.34$, $\mu = 5.0$. In (b)–(d) parameters are $\gamma_1 = 0.34$, $\gamma_2 = 0.37$, and $\mu_{1,2} = 5.0$: y- and z-coupled subsystems [in Eq. (2) $d_1 = d_2 = d$]. (b) The four largest Lyapunov exponents. (c) Difference of phase variables $\phi_1 - \phi_2$ for nonsynchronous ($d = 0.007; 0.0078; 0.008$) and synchronous ($d = 0.0085$) regimes. (d) The mean frequency ratio Ω_1/Ω_2 vs coupling.

scales. In order to achieve synchronization, it is obviously necessary to have synchronization of both subtypes of behavior: rotations and oscillations. As our numerical simulations show, the occurrence of RCPS is possible only for a very small parameter mismatch between both the subsystems in Eq. (2). If the parameter mismatch is large enough, GCPS and GCS set in simultaneously (Fig. 4), or GCS occurs before GCPS (Fig. 5).

Let us first consider the case when GCPS and GCS are achieved at the same critical coupling. We chose parameters ($\gamma_1 = 0.34$, $\gamma_2 = 0.37$, and $\mu_{1,2} = 5.0$) in such a way that the noncoherence of motions in both the subsystems in Eq. (2) is very high. So we have $D_{\phi_1} \approx 0.94$ and $D_{\phi_2} \approx 1.084$. In Fig. 4 we show, as before, the four largest Lyapunov exponents

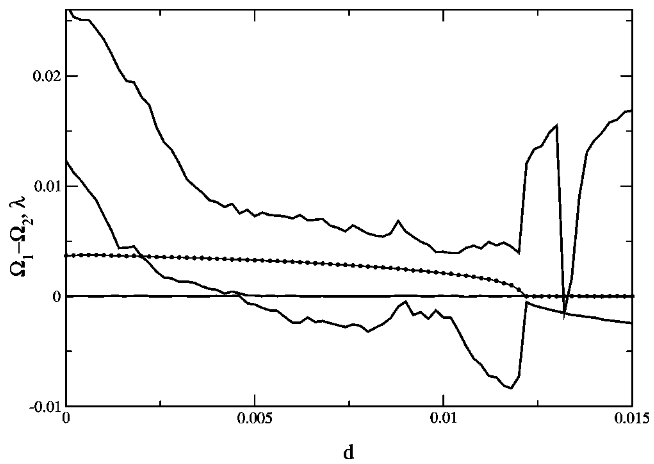

FIG. 5. Synchronization of oscillatory-rotatory phase variables. GCS occurs before GCPS. Parameters are $\gamma_1 = 0.34$, $\gamma_2 = 0.39$, and $\mu_{1,2} = 5.0$: y- and z-coupled subsystems [in Eq. (2) $d_1 = d_2 = d$]. The three largest Lyapunov exponents and the mean frequency difference $\Omega_1 - \Omega_2$ (circles) vs coupling are shown.

[Fig. 4(b)] and the mean frequencies ratio [Fig. 4(d)] versus coupling, as well as the difference between phase variables $\phi_1 - \phi_2$ for different coupling strengths [Fig. 4(c)]. These figures indicate that the transitions to GCPS and GCS occur at $d \approx 0.0082$.

Contrary to the presented examples, where with increase of coupling phase synchronization sets in before or simultaneously with generalized synchronization, we will show the possibility that phase synchronization emerges after the generalized one [14]. We take a relatively large parameter mismatch ($\gamma_1 = 0.34$, $\gamma_2 = 0.39$, and $\mu_{1,2} = 5.0$). In Fig. 5 we plot the mean frequency difference $\Omega_1 - \Omega_2$ and the three largest Lyapunov exponents. One of the positive Lyapunov exponents, λ_2, becomes negative at $d \approx 0.0046$, i.e., generalized synchronization sets in. But conditions (5) and (6) for frequency and phase locking are fulfilled only beyond $d \approx 0.012$. Therefore, generalized synchronization is weaker than phase synchronization in this case. The Lyapunov exponent λ_2 demonstrates an interesting feature. It increases rapidly and almost jumps to zero (but does not reach it), if the coupling is close to the critical value d corresponding to the transition to GCPS.

We have to note that if the noncoherence properties are very large phase synchronization cannot be achieved for any coupling strength.

VI. HARD AND SOFT TRANSITIONS TO PHASE SYNCHRONIZATION

We have found that phase synchronization of two coupled systems (2) can appear or vanish in two ways: soft and hard transition. The soft transition described in all examples in the preceding sections is characterized through a smooth locking of the observed frequencies. Also the topological changes in the phase space appear smoothly. But for the hard transition

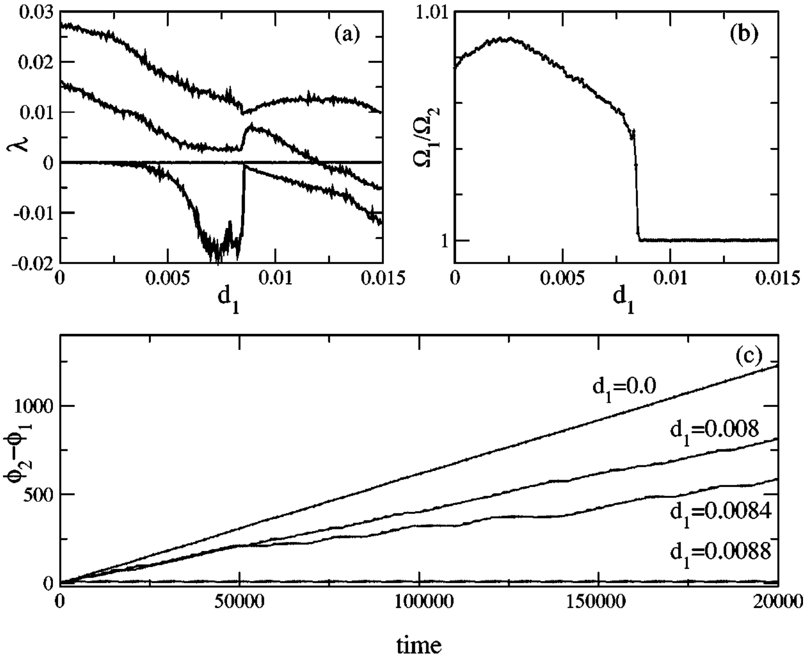

FIG. 6. Hard transition to RCPS. Parameters are $\gamma_1=0.645$, $\gamma_2=0.636$, $\mu_1=3.0$, $\mu_2=3.05$, and $d_2=0$. (a) The four largest Lyapunov exponents. (b) The mean frequency ratio Ω_1/Ω_2 vs coupling. (c) Difference of phase variables $\phi_2-\phi_1$ for nonsynchronous ($d_1=0.0;0.008;0.0084$) and synchronous ($d_1=0.0088$) regimes. At chosen parameter values $D_{\phi_1}\approx 0.219$ and $D_{\phi_2}\approx 0.218$.

to phase and frequency locking quite another situation takes place. Such a transition is illustrated by Fig. 6, where we plot as before the four largest Lyapunov exponents [Fig. 6(a)] and the mean frequencies ratio [Fig. 6(b)] versus coupling, as well as the difference between phase variables $\phi_1-\phi_2$ for different coupling strengths [Fig. 6(c)]. In Fig. 7 the projections of the trajectories of system (2) on planes (ϕ_1,ϕ_2) [Figs. 7(a) and 7(b)] and (y_1,y_2) [Figs. 7(c) and 7(d)] are presented.

The relatively large jump in the mean frequency ratio $\rho=\Omega_1/\Omega_2$ from nonsynchronous ($\rho\neq 1$) to synchronous ($\rho=1$) hyperchaotic behavior can be considered as a manifestation of a hard transition to phase synchronization. Indeed, for very small changes in the coupling, strong changes in the phase difference evolution [Fig. 6(c)] and in the phase portrait (Fig. 7) are observed. For $d_1=0.0084$, i.e., when d_1 is very close to the critical value d_1^1, only very short intervals of synchronization episodes are observed in the phase difference (compare with Figs. 1(c), 2(c), and 4(c) that demonstrate phase differences for the oscillatory case where the transition to phase synchronization is soft). The projections of the hyperchaotic attractor on planes (ϕ_1,ϕ_2) and (y_1,y_2) before and after the transition to phase synchronization are presented in Fig. 7. For the synchronous regime the chaotic

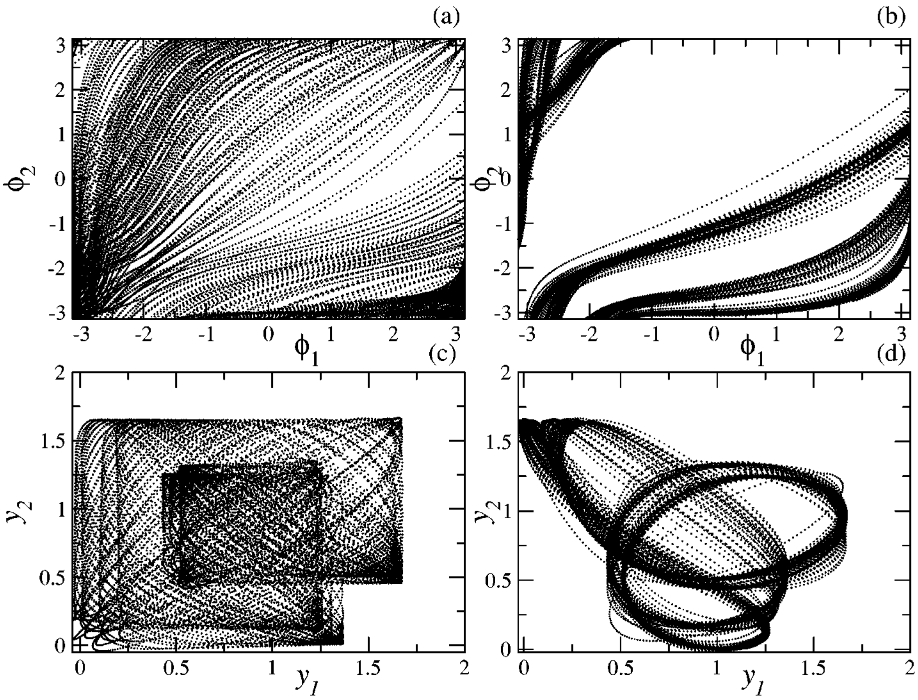

FIG. 7. Projections of the trajectories of system (2) on the planes (ϕ_1,ϕ_2) [(a) and (b)] and (y_1,y_2) [(c) and (d)] for $\gamma_1=0.645$, $\gamma_2=0.636$, $\mu_1=3.0$, $\mu_2=3.05$, and $d_2=0$ outside the synchronization region [(a) and (c)] ($d_1=0.0084$) and within the synchronization region [(b) and (d)] ($d_1=0.0088$).

trajectory lies within relatively narrow bands in the phase space [Figs. 7(b) and 7(d)], while when synchronization is lost these bands smear and merge together [Figs. 7(a) and 7(c)]. Such a hard transition to a band-structured attractor can be explained as follows. In Ref. [28] it was shown that chaotic phase synchronization takes place in the parameter region where all unstable periodic orbits, embedded in the chaotic attractors, are synchronized. For the presented case, the hard transition to phase synchronization is caused by the fact that boundaries of the Arnold tongues corresponding to synchronization of unstable orbits are very close to each other. Another interesting result similar to that presented in Fig. 5 can be seen in Fig. 6(a). When the coupling increases, one of the zero Lyapunov exponents initially remains equal to zero, then it becomes negative and jumps to zero, without reaching it. This happens when the coupling is close to the critical value d_1^1 corresponding to the transition to RCPS, and then beyond d_1^1 this Lyapunov exponent decreases again.

VII. CONCLUSIONS

We have found that rotatory, oscillatory and oscillatory-rotatory synchronization can occur in two coupled autonomous chaotic phase systems. Three types of synchronization have been studied: (i) Real chaotic phase synchronization (RCPS), which is a synchronization occurring when two Lyapunov exponents are positive and when frequency and phase locking conditions are fulfilled; (ii) generalized chaotic phase synchronization (GCPS), a synchronization occurring when only one Lyapunov exponent is positive and when frequency and phase locking conditions are fulfilled; and (iii) generalized synchronization (GCS), a synchronization occurring when only one Lyapunov exponent is positive and when frequency and phase locking conditions are not fulfilled.

Depending on the coherence properties of the motions, which can be measured by the diffusion of the phase variable, we observe four transitions to phase synchronization. For small diffusion, the onset of phase synchronization is accompanied by the change of the Lyapunov spectrum (one of the zero Lyapunov exponents becomes negative shortly before the onset). If the diffusion of the phase variable is strong, then phase and generalized synchronization (one of the positive Lyapunov exponents becomes negative shortly before) occur simultaneously, or generalized synchronization sets in before phase synchronization. For intermediate diffusion, phase synchronization appears via an interior crises of the hyperchaotic set.

Obtained results show that topological (e.g., coherence) properties of motion play a crucial role in the appearance of chaotic phase and generalized synchronization. Complication in the topological structure of motions, e.g., an increase in the noncoherence caused by the change of system parameters is observed for many dynamical systems. For example, the change of control parameters in the Rössler oscillator leads to the transition from a phase-coherent to ''funnel'' chaotic attractor. Appearance of phase synchronization of phase-coherent attractors in coupled Rössler oscillators, first demonstrated in Ref. [9], is quite similar to the appearance of synchronization of oscillatory phase variables described above (Sec. IV). Recently [29], the onset of chaotic phase synchronization was observed for coupled funnel attractors in coupled Rössler oscillators. It occurs via an interior crisis of hyperchaotic set as in the case of synchronization of rotatory phase variables presented in our paper in Sec. III. Therefore, the results presented in our paper seem to be typical for coupled chaotic oscillators.

Our results are of special importance from the points of view of phase locking effects in coupled Josephson junctions and in the theory of automatic synchronization. For example, the onset of chaotic synchronization of phase variables of two standard PLL circuits with simplest second-order filter can be used in secure communications based on the effect of chaotic synchronization. Synchronization phenomena observed in Eq. (2) can be very easily reproduced in the experimental testing. Equations (2) describe dynamics of two chaotic PLL coupled through an additional frequency discriminator. The possible variants of concrete electronics schemes corresponding to Eq. (2) can be found in Ref. [30]. An obvious advantage of proposed synchronization schemes is in the fact, that the chaotic phase synchronization can be obtained for very small coupling. For example, in Ref. [27] the coupling strengths needed to achieve almost complete chaotic synchronization are $\approx 50-100$ times larger than in the presented case. Extension of obtained effects to the networks of coupled PLL's and Josephson junction should be a subject of future experimental and theoretical works.

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$$\frac{\hbar}{2er}\dot{\phi}+I\sin\phi+\dot{Q}=I_B,\quad L\ddot{Q}+R\dot{Q}+\frac{1}{C}Q=0,$$

where ϕ is the wave-function phase difference across the Josephson junction, r is the junction resistance, I is the junction critical current, I_B is the constant biased current, Q is the charge on the load capacitor, $\hbar$ is Planck's constant divided by 2π, e is the elementary charge, and L,R,C are loaded inductance, resistance, and capacitance, respectively. If $LI,RI\ll 1$, the above equation can be rewritten as an ordinary differential equation of the third order:

$$\frac{L\hbar}{2er}\frac{d^3\phi}{dt^3}+\frac{R\hbar}{2er}\frac{d^2\phi}{dt^2}-\frac{I_B}{C}+\frac{\hbar}{2erC}\frac{d\phi}{dt}+\frac{I}{C}\sin\phi=0,$$

which after some parameter substitutions can be easily reduced to Eq. (1).
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